n-P Scattering at low energies / Partial wave analysis of n-P scattering ①

In order to avoid complications due to coulomb forces we shall consider the scattering of neutrons by free protons i.e those not bounded to molecules. However in practice the protons are of course bound to molecules but the molecular binding energy is only about 0.1 ev. Therefore if the incident neutrons have an energy greater than about 1ev, the proton can be regarded as free.

In describing elastic scattering events like the scattering of neutrons by free protons, it is more convenient to use the centre of mass system. For two particles of equal mass such as the (n-P) system, the reduced mass is equal to half the mass of either. Let us suppose that the neutron and proton interact via a spherically symmetric force field whose potential function is $V(r)$, where r is the distance between the particles.

The Schrodinger equation for a central potential $V(r)$ in the centre of mass system for the n-P system is

$$\left[\nabla^2 + \frac{M}{\hbar^2}\{E - V(r)\}\right]\psi = 0; \quad \text{——— (1)}$$

where M is the reduced mass of the n-P system

For scattering, the boundary condition is that at large distances from the scattering centre, the wave should be made up of two parts: (i) the incident plane wave that describes the unscattered particles and superimposed upon it. (ii). an outgoing scattered spherical wave which emanates from the scattering centre.

Therefore, we seek a solution of equation (1) in the asymptotic form.

Therefore, $\psi = \psi_{inc} + \psi_{sc}$; ——— (2)

The wave function that describes an incident plane wave (a beam of particles) moving in the positive z-direction is

$$\psi_{inc} = e^{ikz} = e^{ikr\cos\theta} \quad \text{——— (3)}$$

where $K = \sqrt{\frac{ME}{\hbar^2}}$

which is a solution of the wave equation (1) with $V(r)$ is set equal to zero; then (1) becomes

$$\left[\nabla^2 + \frac{2mE}{\hbar^2}\right]\psi_{inc} = 0.$$

Here setting $V(r) = 0$ indicates to switching off the scattering potential and thereby eliminate scattering so that the total wave function (ψ) becomes identical with the incident wavefunction ψ_{inc}.

The suitable form for the scattered wave function is:

$$\psi_{sc} = f(\theta)\frac{e^{ikr}}{r}; \quad \text{---} (4)$$

for large 'r', $f(\theta)$ in this expression refers to the amplitude of the scattered wave in the direction θ.

Therefore, the wave function ψ can be written as—

$$\psi = \psi_{inc} + \psi_{sc} = e^{ikz} + f(\theta)\frac{e^{ikr}}{r}. \quad \text{---} (5)$$

Expanding the incident plane wave function e^{ikz} in terms of Legendre polynomials $P_l(\cos\theta)$

$$\psi_{inc} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} \cancel{B_l(r) P_l(\cos\theta)}; \quad \text{---} \cancel{(6)}$$

where l is an integer representing the various partial waves.

$$= \sum_{l=0}^{\infty} B_l(r)\, P_l(\cos\theta) \quad ; \text{---} (6)$$

The radial functions $B_l(r)$ appearing in equation (6) is given by

$$B_l(r) = i^l (2l+1)\, j_l(kr) \quad \text{---} (7)$$

where $j_l(r)$ is the spherical Bessel function which is related to the ordinary Bessel function through the formula

$$j_l(kr) = \left(\frac{\pi}{2kr}\right)^{1/2} J_{l+\frac{1}{2}}(kr) \quad ; \text{---} (8)$$

and can be represented as

$$j_l(kr) = (-kr)^l \left[\frac{1}{kr}\frac{d}{d(kr)}\right]^l_0 \left(\frac{\sin kr}{kr}\right) \quad \text{---} (9)$$

whence asymptotically

$$j_l(kr)_{r\to\infty} \to \frac{\sin(kr - \frac{l\pi}{2})}{kr} \quad ; \text{---} (10)$$

∴ Asymptotically; $B_l(r)$ from (7) are given by

$$B_l(r)_{r\to\infty} = i^l (2l+1)\frac{\sin(kr - \frac{l\pi}{2})}{kr} \quad \text{---} (11a)$$

$$= \frac{1}{2ikr} i^l (2l+1)\left[e^{i(kr - l\pi/2)} - e^{-i(kr - l\pi/2)}\right] \quad \text{---} (11)$$

The spherical Bessel functions $j_l(kr)$ for various values of l are given below

$$j_0(kr) = \frac{\sin kr}{kr}$$

$$j_1(kr) = \frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{(kr)}$$

$$j_2(kr) = \left[\frac{3}{(kr)^3} - \frac{1}{kr}\right]\sin(kr) - \frac{3\cos(kr)}{(kr)^2}$$

Similarly $f(\theta)$ may also be expanded in terms of the Legendre Polynomials as follows:

$$f(\theta) = \frac{i}{2K}\sum_{l=0}^{\infty} f_l\,(2l+1)\,P_l(\cos\theta)\,; \quad (12)$$

Substituting the values from equations (6), (7) and (12) in equation (5) we get

$$\psi = \psi_{inc} + \psi_{sc} = \sum_{l=0}^{\infty}\left[i^l(2l+1)\,j_l(kr) + f_l\,\frac{e^{ikr}}{r}\right]P_l(\cos\theta)\,; \quad (13)$$

At lower energies, below 10 Mev, most of the scattering is due to $l=0$ particles i.e the number of partial waves is severely limited in this case and demands to study the scattering only for $l=0$ i.e S-wave.

From equation (11) for $l=0$

$$B_0(r) = \frac{\sin(kr)}{kr} = 1 - \frac{(kr)^2}{6} + \cdots \quad (14a)$$

and for $l=1$

$$B_1(r) = 3i\left[\frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr}\right]$$

$$= 3i\left[\frac{kr}{3} - \frac{(kr)^3}{30} + \cdots\right]; \quad (14b)$$

$$\therefore \left|\frac{B_1(r)}{B_0(r)}\right|^2 \simeq (kr)^2\,; \quad (14c)$$

Considering a neutron of energy 1 Mev in the L-system, it will be 0.5 Mev in the C-M system. Neutron momentum then is

$$P = (2ME)^{1/2} = \left[\frac{2\times1.67\times10^{-27}\,kg\times(1.6\times10^{-13}\,J)}{2}\right]^{1/2}$$

$$= 1.63\times10^{-20}\ kg.\,m.\,sec^{-1}.$$

and its wave number

$$K = \frac{P}{\hbar} = \frac{1.63\times10^{-20}\ kg.\,m.\,sec^{-1}}{1.0545\times10^{-34}\,J.\,sec} \simeq 1.55\times10^{14}\,m^{-1}$$

If we assume the nuclear forces to have a range $r_0 = 2$ Fermi, then

$$\left|\frac{B_1(r)}{B_0(r)}\right|^2 \simeq (kr_0)^2 \simeq \left[(1.55\times10^{14}\,m^{-1})\times(2\times10^{-15}\,m)\right]^2$$

$$\simeq (0.31)^2 \simeq 0.091$$

i.e. at an energy of 1 Mev only about 9% of the scattering

(4) is due to neutrons with $l=1$. At an energy of 10 MeV, the percentage of scattering for neutron is increased to 1%. Therefore in the energy range below 10 MeV, S-wave scattering ($l=0$) is predominant.

Substituting $l=0$ in equation (6) gives the undisturbed first partial wave.

$$\psi_{inc}^{l=0} = B_0(r)\, P_0(\cos\theta) \simeq \frac{\sin(kr)}{kr} \quad [\, P_0(\cos\theta)=1$$

$$= \frac{e^{ikr} - e^{-ikr}}{2ikr} \; ; \qquad (15)$$

The only other possible change in the wave can be change of phase let us try to study the effect of this phase change on the total wave function outside the range of the scattering potential. The phase shift experienced by the outgoing wave be denoted by $2\delta_0$. For $l=0$ part of the plane wave, instead of (15) we get

$$\psi_0 = \frac{e^{i(kr+2\delta_0)} - e^{-ikr}}{2ikr} = \frac{e^{i\delta_0}\cdot[e^{i(kr+\delta_0)} - e^{-i(kr+\delta_0)}]}{2ikr}$$

$$= e^{i\delta_0}\, \frac{\sin(kr+\delta_0)}{kr} \qquad (16)$$

The scattered wave function $\psi_{sc}^{l=0}$ can be obtained from equation (5).

$$\psi_{sc}^{l=0} = \psi_0 - \psi_{inc}^{l=0}$$

Substituting the value of ψ_0 and $\psi_{inc}^{l=0}$ from equations (16) and (15) we get

$$\psi_{sc}^{l=0} = \frac{e^{i(kr+2\delta_0)} - e^{-ikr}}{2ikr} - \frac{e^{ikr} - e^{-ikr}}{2ikr}$$

$$= \frac{e^{i(kr+2\delta_0)} - e^{ikr}}{2ikr}$$

$$= \frac{e^{ikr}}{2ikr}\left[e^{2i\delta_0} - 1\right] ; \qquad (17)$$

Note that $\psi_{sc}^{l=0}$ contains only outgoing terms and disappears as soon as we 'switch off' the scattering potential i.e. putting $\delta=0$.

Comparing equations (4) and (17) we get

$$[f(\theta)]_{l=0} = \frac{e^{2i\delta_0} - 1}{2ik} = \frac{e^{i\delta_0}[e^{i\delta_0} - e^{-i\delta_0}]}{2ik}$$

$$= \frac{e^{i\delta_0}\sin\delta_0}{k} \qquad (18)$$

where $[f(\theta)]_{l=0}$ is the amplitude of the scattered wave for $l=0$ wave or an S-wave.

(2) The Cross-section $\sigma(\theta)\,d\Omega$ for the scattering into the solid angle $d\Omega$ is defined as

$$\sigma(\theta)\,d\Omega = \frac{N_{sc}\,d\Omega}{\text{Incident flux}}$$

where $N_{sc}\,d\Omega \rightarrow$ No. of particles scattered through a large sphere of radius r into the solid angle $d\Omega = \sin\theta\,d\theta\,d\phi$

and $\sigma(\theta)$ is the differential scattering Cross-section.

N_{sc} = Probability Current density X area

$$\therefore N_{sc}\,d\Omega = \frac{\hbar}{2iM}\left(\psi_{sc}^{*}\frac{\partial\psi_{sc}}{\partial r} - \psi_{sc}\frac{\partial\psi_{sc}^{*}}{\partial r}\right) r^2\,d\Omega$$

where $\frac{\hbar}{2iM}\left(\psi_{sc}^{*}\frac{\partial\psi_{sc}}{\partial r} - \psi_{sc}\frac{\partial\psi_{sc}^{*}}{\partial r}\right)$ is the probability Current density of the scattered particles and $d\Omega = \frac{dA}{r^2}$

$$\therefore N_{sc}\,d\Omega = \frac{\hbar}{iM}\left(\psi_{sc}^{*}\frac{\partial\psi_{sc}}{\partial r}\right) r^2\,d\Omega$$

$$= \frac{\hbar}{iM}\,\mathrm{Im}\left(\psi_{sc}^{*}\frac{\partial\psi_{sc}}{\partial r}\right) r^2\,d\Omega$$ where Im stands for imaginary part of the term.

Now since $\psi_{sc} = f(\theta)\,\frac{e^{ikr}}{r}$

$$\therefore N_{sc}\,d\Omega = \frac{\hbar}{iM} f^{*}(\theta)\,\frac{e^{-ikr}}{r}\,\frac{\partial}{\partial r}\left[f(\theta)\,\frac{e^{ikr}}{r}\right] r^2\,d\Omega$$

$$= \frac{\hbar}{iM}\cdot\frac{ik}{r^2} f^{*}(\theta)\,f(\theta)\,r^2\,d\Omega$$

$$= \frac{\hbar k}{M}\,|f(\theta)|^2\,d\Omega; \quad (19)$$

And the incident flux $= \frac{\hbar}{iM}\,e^{-ikz}\frac{\partial}{\partial z}\left(e^{ikz}\right)d\Omega$

since $\psi_{inc} = e^{ikz}$

$$= \frac{\hbar}{iM}\,e^{-ikz}\cdot(ik)\cdot e^{ikz}\,d\Omega$$

$$= \frac{\hbar k}{M}\,d\Omega = v\,d\Omega$$

$$\therefore \sigma(\theta)\,d\Omega = \frac{N_{sc}\,d\Omega}{v\,d\Omega} = \frac{\frac{\hbar k}{M}|f(\theta)|^2\,d\Omega}{v\,d\Omega}; \quad (20)$$

$$\text{or } \sigma(\theta) = |f(\theta)|^2; \quad (21)$$

With the aid of equation (18) the differential scattering cross-section for an S wave ($l=0$) is given by.

$$\sigma(\theta) = \left[|f(\theta)|^2 \right]_{l=0} = \frac{e^{2i\delta_0} \cdot \sin^2\delta_0}{k^2}$$

$$\boxed{\sigma(\theta) = \frac{\sin^2\delta_0}{k^2}} \quad \text{Since } e^{2i\delta_0} = 1 \qquad (22)$$

And the total scattering cross-section for an s wave is

$$\sigma_{sc,0} = \int \sigma(\theta)\, d\Omega = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{\sin^2\delta_0}{k^2} \sin\theta\, d\theta\, d\phi$$

$$= 2\pi \int_{\theta=0}^{\pi} \frac{\sin^2\delta_0}{k^2} \sin\theta\, d\theta.$$

$$\boxed{\sigma_{sc,0} = 4\pi \frac{\sin^2\delta_0}{k^2} = 4\pi \lambda^2 \sin^2\delta_0} \qquad (23)$$

where $\lambda = \frac{1}{k}$

Equations (22) and (23) giving the differential scattering cross-section and total scattering cross-section respectively for $l=0$ neutron